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ON THE COHOMOLOGY OF ERGODIC GROUP ACTIONS

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ABSTRACT
We consider three problems concerning cocycles of ergodic group actions: behavior of cohomology under the restriction of an ergodic semi-simple Lie group action to a lattice subgroup; "compactness" of bounded cocyles; and the relation of relative almost periodicity to relative discrete spectrum for extensions of ergodic actions.

1. Introduction

If G is a separable locally compact group acting ergodically on a standard measure space (S,μ) a cocycle with values in a group M is a Borel function $\alpha: S\times G\to M$ such that $\alpha(s,gh)=\alpha(s,g)\alpha(sg,h)$ almost everywhere, for each $g,h\in G$. These are precisely the functions that enable one to define actions on bundles or sections of bundles over S where the fiber is a space on which M acts. In this paper we present a collection of three results concerning cocycles which are essentially complements to previous work of the author on problems in ergodic theory in which the study of various classes of cocycles played a sgnificant role. We refer the reader to [7], [11], [15] as background and general references for the types of facts concerning cocycles we will need.

The first problem we consider is the following. Suppose that G is a finite product of connected simple Lie groups with finite center and that $\Gamma\subset G$ is an irreducible lattice subgroup. If G acts ergodically on S, one knows precisely when the restriction of the action to Γ is ergodic. C. C. Moore [5] has shown that if G is transitive on S then Γ will be ergodic if and only if the stabilizers of S are non-compact, and the author [15] has shown that Γ will be ergodic on S for any properly ergodic G-space S. If Γ is ergodic, it is a natural question to examine the relationship between the G-action and the Γ-action. For example, it follows

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from [18, proposition 3.4] that the G-action is amenable in the sense of [16] if and only if the Γ-action is amenable. Our aim in Section 2 below is to examine how cohomology behaves under the restriction of the G-action to Γ. More precisely, if we let $H^1(S \times G; M)$ be the cohomology classes of cocycles with values in M, then we have a natural map $H^1(S \times G; M) \to H^1(S \times \Gamma; M)$ which is a homomorphism of groups if M is abelian. We show that this map is injective if M is either discrete or abelian, and that for arbitrary locally compact M that the inverse image of the identity consists of the identity alone.

In Section 3 we present a general fact on bounded cocycles of an arbitrary ergodic action. More precisely, if $\alpha : S \times G \to M$ is bounded in the sense that $\{\alpha(s, g) \mid g \in G\}$ has compact closure for almost all s, then α is equivalent to a cocycle into a compact subgroup of M. This result has been previously shown for $M = Z$ or $\mathbf{R}$ in [8] and [2, II, theorem 6], and for M a closed subgroup of $\mathrm{GL}(n)$ in [14].

Finally, in Section 4, we present a result concerning unitary cocycles, i.e., cocycles taking values in the unitary group of a Hilbert space. One can try to decompose such a cocycle into irreducible subcocycles just as for unitary representations, and we say that the cocycle has discrete spectrum if it is a direct sum of finite dimensional irreducible subcocycles. This notion plays a central role in the author's measure theoretic analogue of the Furstenberg structure theorem [11], [12]. It is well known that a criterion for a representation of a group to have discrete spectrum can be formulated in terms of almost periodic vectors, i.e., vectors whose orbits are precompact in the norm topology. We shall present a similar criterion for a cocycle to have discrete spectrum in terms of vectors which are defined by a suitable compactness condition. This was actually carried out for integer actions with finite invariant measure in [14] by a technical argument involving the ergodic theorem. Here, although our result is probably true in general, we shall require that S be an amenable G-space in the sense of [16]. This includes all ergodic actions of amenable groups and many others as well.

2. Cohomology and the restriction of semisimple Lie group actions to lattices

We recall with somewhat more precision the basic definitions. By an ergodic G-space, where G is a locally compact second countable group, we mean a standard Borel space S, together with a Borel G-action $S \times G \to S$, and a probability measure μ on S that is quasi-invariant and ergodic under G. A Borel function $\alpha : S \times G \rightarrow M$, where M is a standard Borel group, is called a cocycle with values in M if for all $g, h \in G$,

$$\alpha(s, gh) = \alpha(s, g)\alpha(sg, h)$$

for almost all s. If $\alpha, \beta : S \times G \rightarrow M$ are cocycles, they are called equivalent or cohomologous if there is a Borel function $\varphi : S \rightarrow M$ such that for each g,

$$\varphi(s)\alpha(s, g)\varphi(sg)^{-1} = \beta(s, g) \qquad \text{a.e.}$$

The set of equivalence classes of cocycles with values in M will be denoted by $H^1(S \times G; M)$. In general, this is a set with a distinguished element $e \in H^1(S \times G; M)$ corresponding to the trivial cocycle (i.e., constantly the identity of M), but if M is abelian, $H^1(S \times G; M)$ has the structure of an abelian group under pointwise multiplication.

Suppose now that G is a finite product of connected non-compact simple Lie groups with finite center and that Γ is an irreducible lattice subgroup [6]. Let S be an ergodic G-space on which the restriction of the action to Γ is still ergodic. As recalled in the introduction, this will be the case except for $S = G/K$, where $K \subset G$ is compact [5], [15]. If $\alpha : S \times G \rightarrow M$ is a cocycle, we denote by α_Γ the restriction of α to $S \times \Gamma$. The main point of this section is to prove the following result.

THEOREM 2.1. *With G, Γ, S as above, the map $H^1(S \times G; M) \rightarrow H^1(S \times \Gamma; M)$, $\alpha \rightarrow \alpha_\Gamma$, is injective if M is either abelian or discrete. For general locally compact M, α_Γ is trivial if and only if α is trivial.*

We begin with the following general observation which enables us to transform the question concerning the restriction of the cocycle to the Γ-action to a question concerning the "restriction" of a cocycle to another G-action. We recall that since Γ is ergodic on S, G is ergodic on the product space $S \times G/\Gamma$ [15, theorem 4.2]. For $\alpha \in H^1(S \times G; M)$, we define $\tilde{\alpha} \in H^1(S \times G/\Gamma \times G; M)$ by $\tilde{\alpha}(s, x, g) = \alpha(s, g)$. Then $\tilde{\alpha}$ is called the restriction of α to $S \times G/\Gamma$.

PROPOSITION 2.2. *There is an identity preserving bijection $\Phi : H^1(S \times \Gamma; M) \rightarrow H^1(S \times G/\Gamma \times G; M)$ such that for $\alpha \in H^1(S \times G; M)$, we have $\Phi(\alpha_\Gamma) = \tilde{\alpha}$. If M is abelian, Φ is a group homomorphism.*

PROOF. This result follows from the fact that $S \times \Gamma$ and $S \times G/\Gamma \times G$ are similar measure groupoids [3], [7]. However, we shall present a sketch of a proof which does not explicitly require the measure groupoid machinery, leaving verification of details to the reader. Choose a Borel function $\theta : G/H \to G$ such that for all $g \in G$, $g\theta(p(g)) \in H$ where $p : G \to G/H$ is the projection. If $\beta : S \times H \to M$ is a cocycle, define $\beta_0 : S \times G/H \times G \to M$ by

$$\beta_0(s, x, g) = \beta(s\theta(x), \theta(x)^{-1} g\theta(xg)).$$

One readily checks that β_0 is a cocycle and that if α and β are equivalent with the equivalence implemented by $\varphi : S \to M$, then α_0 and β_0 are equivalent with the equivalence implemented by $\psi(s, x) = \varphi(s\theta(x))$. To construct the inverse map, suppose $\gamma : S \times G/H \times G \to M$ is a cocycle. Changing γ on a null set, we can find an inessential contraction on which γ is strict [7]. (See [16, section 3] for a formulation in the present framework.) Then for almost all $x \in G/H$, $\gamma_x : S \times \Gamma \to M$ defined by

$$\gamma_x(s, h) = \gamma(sg, x, g^{-1}hg),$$

where $[e] \cdot g = x$, will define a cocycle, and one can check that on a conull set of x, these cocycles are equivalent. This defines an element $[\gamma_1] \in H^1(S \times \Gamma; M)$ which is independent of the choices involved. One then checks in a somewhat tedious but straightforward manner that $\beta \to \beta_0$ and $\gamma \to \gamma_1$ are inverse maps, and that the remaining assertions of the proposition are true.

We now prove Theorem 2.1 for M compact.

LEMMA 2.3. *If M is compact and α_Γ is trivial, then α is trivial.*

PROOF. Since M is compact, we can assume, by passing to an equivalent cocycle, that $S \times_\alpha M$ is ergodic [11, corollary 3.8]. If S is properly ergodic, $S \times_\alpha M$ will be as well, and so Γ is ergodic on $S \times_\alpha M$, which is just the Γ-space $S \times_{\alpha_\Gamma} M$. It follows that α_Γ cannot be trivial. On the other hand, if S is essentially transitive, say $S = G/H$, then α corresponds to a homomorphism $h : H \to M$ with dense range [10, theorem 8.27]. If $h(H) = M$, then G will be essentially transitive on $S \times_\alpha M$ and the stabilizer in G of this space will be the kernel of h. Since M is compact and G is not, $\ker(h)$ is not compact, so by Moore's Theorem, we again have that Γ is ergodic on $S \times_\alpha M$, and hence α_Γ is not trivial. Finally, if $h(H)$ is dense but not closed in M, G will be properly ergodic on $S \times_\alpha M$, and another application of [15] ensures that Γ is ergodic on $S \times_\alpha M$, and hence α_Γ is non-trivial.

To prove non-triviality of α_Γ for a general locally compact M, we use the following fact.

LEMMA 2.4. *Let M be a locally compact second countable group, and (Ω_M, ν) be a completion of $(L^2(M, \mathbf{R}), m)$ where m is the canonical finitely additive Gaussian measure on $L^2(M, \mathbf{R})$. Then M acts in a natural way on Ω_M, and so we can form the skew product G-action $S \times_\alpha \Omega_M$. Then α is equivalent to a cocycle into a compact subgroup of M if and only if $S \times_\alpha \Omega_M$ is not ergodic.*

PROOF. Suppose α is equivalent to a cocycle β into a compact subgroup K. Then $\pi(K)$ leaves a finite dimensional subspace of $L^2(M) \ominus \mathbf{C}$ invariant where π is the regular representation of M. It follows from [9] that K is not ergodic on Ω_M, and so $S \times_\beta \Omega_M$, and hence $S \times_\alpha \Omega_M$, is not ergodic. Conversely, suppose $S \times_\alpha \Omega_M$ is not ergodic. Then there is a Borel map $\varphi : S \to B$, where B is the unit ball in $L^2(\Omega_M) \ominus \mathbf{C}$, such that

$$U(\alpha(s,g))\varphi(sg) = \varphi(s) \qquad \text{for each } g \text{ and almost all } s,$$

where U is the unitary representation of M on $L^2(\Omega_M)$ induced by the action on Ω_M. The action of M on B is smooth, in that the quotient Borel structure is standard [17, proof of theorem 3.1], [1], and the above equation says $\varphi(sg) \equiv \varphi(s)$ for each g and almost all s. By ergodicity, $\varphi(s)$ will all be in the same orbit for a conull set of s. Let $h \in B$ be a point in this orbit, and choose a Borel map $\theta : M \cdot h \to M$ such that $U(\theta(h'))h' = h$ for all $h' \in M \cdot h$. Let $\psi(s) = \theta(\varphi(s))$, and define $\beta \sim \alpha$ by

$$\beta(s,g) = \psi(s)\alpha(s,g)\psi(sg)^{-1}.$$

Then for each g and almost all s, we see that $U(\beta(s,g))h = h$, i.e., $\beta(s,g)$ is in the stabilizer of the vector $h \in B$. But this stabilizer is compact [17, proof of theorem 3.1], completing the proof.

LEMMA 2.5. *If M is locally compact and second countable, and α_Γ is trivial, then α is trivial.*

PROOF. If α is equivalent to a cocycle into a compact subgroup of M, the result follows from Lemma 2.3. Otherwise $S \times_\alpha \Omega_M$ is ergodic, and since M is properly ergodic on Ω_M, G is properly ergodic on $S \times_\alpha \Omega_M$. It follows that Γ is ergodic on $S \times_\alpha \Omega_M = S \times_{\alpha_\Gamma} \Omega_M$. By the preceding lemma α_Γ is not equivalent to a cocycle into a compact subgroup, and in particular is non-trivial.

The proof of the second half of Theorem 2.1 is now complete, which of course implies injectivity for M abelian. It remains to consider the first half of the theorem for M discrete.

LEMMA 2.6. *$H^1(S \times G; M) \to H^1(S \times \Gamma; M)$ is injective for M countable and discrete.*

PROOF. Suppose $\alpha, \beta : S \times G \to M$ and $\tilde{\alpha}, \tilde{\beta} : S \times G/\Gamma \times G \to M$ are equivalent. Then there is a Borel function $\varphi : S \times G/\Gamma \to M$ such that for all g,

$$\varphi(s,x)\alpha(s,g)\varphi((s,x)g)^{-1} = \beta(s,g) \qquad \text{for almost all } (s,x),$$

or equivalently,

$$(*) \qquad \beta(s,g)^{-1}\varphi(s,x)\alpha(s,g)^{-1} = \varphi((s,x)g).$$

For each $y \in M$, let $I_y(s)$ be the characteristic function of $A_y(s) = \{x \in G/\Gamma \mid \varphi(s,x) = y\}$. Let B be the unit ball in $L^2(G/\Gamma)$. Then for each y, $S \to B$, $s \to I_y(s)$ is a Borel function. Equation (*) implies that for each $g \in G$, $y \in M$, and almost all s,

$$(**) \qquad I_y(s) = U_g(I_{\beta(s,g)^{-1}y\alpha(s,g)}(sg)),$$

where U_g is the representation of G induced by the indentity representation of Γ. For each $s \in S$, let $C_s = \max\{\mu(A_y(s)) \mid y \in M\}$ where μ is the G-invariant probability measure on G/Γ. (We note that this maximum must clearly be achieved.) The map $s \to C_s$ is Borel. Furthermore, it follows from (**) that C_s is essentially G-invariant, and hence by ergodicity that C_s is equal to a constant c on a conull set. For $y \in M$, let

$$S_y = \{s \in S \mid \mu(A_y(s)) = c\}$$

and for $s \in S$, let N_s be the cardinality of $\{y \mid s \in S_y\}$. We note that $N_s < \infty$ and furthermore N_s is a Borel function. To see this, observe that

$$\{s \mid N_s \geqq n\} = \bigcup_I \left\{ s \mid \sum_{i=1}^{n} \mu(A_{y_i}(s)) = nc \right\}$$

where $I = \{(y_1, \cdots, y_n) \in M^n \mid y_i \neq y_j \text{ for } i \neq j\}$. It follows from (**) that N_s is essentially G-invariant, and ergodicity implies that $N_s = N$ on a conull Borel subset of S. Using the fact that $s \to I_y(s)$ Borel, one can easily construct N Borel functions $f_1, \cdots, f_N : S \to B$ such that for almost all s, $\{f_1(s), \cdots, f_N(s)\} = \{I_{y_1}(s), \cdots, I_{y_N}(s) \mid y_1, \cdots, y_N \text{ distinct and } \mu(A_{y_i}(s)) = c \text{ for all } i\}$. Let H_s be the N dimensional subspace of $L^2(G/\Gamma)$ spanned by $\{f_i(s)\}$. Once again, (**) implies that the Borel field H_s is G-invariant, i.e., for each g, $U_g H_{sg} = H_s$ for almost all s. In other words, the restriction of U_g to $S \times G$ contains a finite dimensional subcycle. Then [15, corollary 3.5] and [5] imply that the projection of H_s onto

$L^2(G/H)\ominus\mathbf{C}$ must be 0 a.e., i.e., $H_s=\mathbf{C}$. It follows that for almost all s, $\varphi(s,x)$ is constant for almost all x. Thus changing φ on a null set, φ is a function only of s, showing that α and β are cohomologous.

This completes the proof of Theorem 2.1.

3. Bounded cocycles

The aim of this section is to prove the following theorem:

THEOREM 3.1. *Suppose $\alpha: S\times G\to M$ is a cocycle where M is locally compact (and separable), and that for almost all s, $\{\alpha(s,h)\mid h\in G\}$ is bounded, i.e., has compact closure in M. Then α is equivalent to a cocycle into a compact subgroup of M.*

This result then subsumes the various known special cases mentioned in the introduction. Applying the theorem to the Radon–Nikodym cocycle of an ergodic action, it implies that if the Radon–Nikodym derivatives are pointwise bounded (not necessarily uniformly bounded) then there is an equivalent σ-finite invariant measure. In this regard, see [2, p. 351].

PROOF. We suppose first that G is countable, so we can assume that α is a strict cocycle, i.e., that the cocycle identity holds for all s,g,h. M is metrizable by a complete separable metric and hence the space $\mathscr{C}$ of compact subsets of M is a complete separable metric space with the Hausdorff metric. For $s\in S$, let

$$A_s=\{\alpha(s,h)\mid h\in G\}.$$

Since α is a Borel function, it is not difficult to check that the map $S\to\mathscr{C}$, $s\to\bar{A}_s$ is measurable. Furthermore, the cocycle identity implies that for each s,g, $\alpha(s,g)\overline{A_{sg}}=\bar{A}_s$. If $A\in\mathscr{C}$ and $x_n\in M$ with $x_nA\to B\in\mathscr{C}$, we must clearly have a subsequence of x_n converging to some $x\in M$ and hence $xA=B$. In other words, the orbits in $\mathscr{C}$ under the action of M are closed. It follows that the orbit space $\mathscr{C}/M$ is standard with the quotient Borel structure [1]. The equation $\alpha(s,g)\bar{A}_{sg}=\bar{A}_s$ implies $\bar{A}_{sg}\equiv\bar{A}_s$ in $\mathscr{C}/M$, and by ergodicity of the action of G on S, there is an orbit, say $M\cdot B$ in $\mathscr{C}/M$ with $\bar{A}_s\in M\cdot B$ for almost all s. Choosing a Borel section of the natural map $M\to M\cdot B$, we obtain a Borel map $\theta: M\cdot B\to M$ such that for each $D\in M\cdot B$, $\theta(D)\cdot B=D$. Let $\varphi(s)=\theta(\bar{A}_s)$. We claim that for each g and almost all s,

$$\varphi(s)\alpha(s,g)\varphi(sg)^{-1}B = B.$$

The left side of this equation is $\varphi(s)\alpha(s,g)\bar{A}_{sg} = \varphi(s)\bar{A}_s = B$. So α is equivalent to a cocycle β taking values almost everywhere in the stabilizer of B, which must clearly be compact since B is compact. Then changing β on a suitable null set gives the desired cocycle.

If G is not countable, let H be a countable dense subgroup. Let Ω_M be as in Lemma 2.4. If α is bounded, so is $\alpha_H = \alpha \mid S \times H$, and so by the result for countable groups and Lemma 2.4, $S \times_{\alpha_H} \Omega_M$ is not ergodic. But this H action is just the restriction to H of the G-action on $S \times_\alpha \Omega_M$, and since H is dense in G, G cannot be ergodic on $S \times_\alpha \Omega_M$ either. An application of Lemma 2.4 then completes the proof.

4. α-compact vectors

We now consider the situation in which $\alpha : S \times G \to U(H)$ where $U(H)$ is the unitary group of a separable Hilbert space H endowed with the strong operator topology. In this context, cocycles are precisely the functions which enable one to define induced representations. Thus

$$(U^\alpha(g)f)(s) = \alpha(s,g)r(s,g)^{1/2}f(sg)$$

defines a unitary representation of G on $L^2(S;H)$ where $r(s,g)$ is the Radon–Nikodym cocycle of the action (see [7], for example). One can form direct sums of unitary cocycles in the obvious way and the question we wish to consider is when α is equivalent to $\alpha_1 \oplus \alpha_2$ (α_i are then called subcocycles) with $\dim \alpha_1 < \infty$, or more generally, when $\alpha = \Sigma^\oplus \alpha_i$, $\dim \alpha_i < \infty$. This is precisely the situation that arises in the study of extensions of ergodic actions with relatively discrete spectrum [11]. Certain criteria are given in [11], [12], [13], and a compactness criterion was given for measure preserving integer actions in [14]. Here we generalize the latter to amenable ergodic actions in the sense of [16].

Definition 4.1. A Borel function $f : S \to H$ is called α-compact if for almost all s, $\{\alpha(s,g)f(sg) \mid g \in G\}$ is precompact in the norm topology of H.

If S is a point, then f becomes an almost periodic vector under the representation α and, as is well known, this implies the existence of finite dimensional invariant subspaces. We wish to show in our more general situation that the existence of (non-trivial) α-compact vectors implies the existence of finite dimensional subcocycles of α. This is equivalent to the existence of U^α-invariant fields of subspaces of the form $\int^\oplus H_s$, where $H_s \subset H$ is finite dimensional. Although this is in all likelihood true in general, we shall make the assumption that S is an amenable G-space [16]. This will always be true if G is amenable, and is often true more generally. As we shall need this assumption only at one point below, we shall not recall the definition of amenability here, but instead refer the reader to [16].

THEOREM 4.2. *Suppose $\alpha : S \times G \to U(H)$ is a unitary cocycle where S is an amenable G-space. If there is a non-zero α-compact vector, then α has finite dimensional subcocycles. Hence, if there is a collection of α-compact vectors $\{w_i\}$ so that the smallest closed U^α-invariant subspace of $L^2(S;H)$ containing $\{w_i\}$ and closed under multiplication by $L^\infty(S)$ is all of $L^2(S;H)$, then α has discrete spectrum (i.e., is equivalent to a direct sum of finite dimensional cocycles).*

PROOF. In light of the strong continuity of the induced representation U^α, it suffices, by passing to a countable dense subgroup, to prove the theorem for G countable. We first claim that for any α-compact function $w, s \to \|w(s)\|$ is essentially bounded. To see this, define

$$f(s) = \sup\{\|\alpha(s,g)w(sg)\| \mid g \in G\}.$$

Then by the precompactness condition, $f(s) < \infty$ a.e., and $f(s)$ is Borel. If $h \in G$, $f(sh) = \sup\{\|\alpha(sh,g)w(shg)\|\}$. But for almost all s, this equals

$$\sup\{\|\alpha(s,h)^{-1}\alpha(s,hg)w(shg)\| \mid g \in G\} = f(s)$$

since $\alpha(s,h)$ is unitary. By ergodicity, f is constant a.e., and hence essentially bounded. Thus, we can assume $\|w(s)\| < 1$ for all s. Since G is countable, we can also suppose that the precompactness condition on w holds for all s, not just on a conull set. We can further assume that $\|w(s)\|$ is bounded away from 0. To see this, choose $0 < a < 1$ such that $S_0 = \{s \mid \|w(s)\| \geqq a\}$ is not null. For almost all $s \in S - S_0$, there is $g \in G$ such that $sg \in S_0$, by ergodicity. Define $w'(s)$ by $w'(s) = w(s)$ for $s \in S_0$ and $w'(s) = \alpha(s,g)w(sg)$ for $s \in S - S_0$ for some fixed selection of g with $sg \in S_0$. Since G is countable, it is clear that this selection can be performed so that $w'(s)$ will be measurable, and one can readily verify that if w is α-compact, so is w'. Replacing w by w' we can thus assume $\|w(s)\| \geqq a$.

Let B be the unit ball in H which is a compact metric space with the weak (or equivalently, weak-*) topology. For each s, let $A_s \subset B$ be defined by $A_s =$

$\{\alpha(s,g)w(sg) \mid g \in G\}^{-}$. The technique of [16, part (ii) of the proof of Lemma 1.7] shows that $s \to A_s$ is a Borel field of compact subsets of B in that $\{(s,x) \mid x \in A_s\}$ is a Borel subset of $S \times B$. Furthermore, we can easily see that A_s is α-invariant, which means $\alpha(s,h)A_{sh} = A_s$ a.e. We also note that $0 \notin A_s$, since $\|w(s)\|$ is bounded away from 0. Let $M(B)$ be the space of probability measures on B. Define $\beta(s,g): C(B) \to C(B)$ to be the adjoint mapping of $\alpha(s,g)^{-1}: B \to B$. Then $\beta(s,g)$ is a Borel cocycle into the isometric isomorphism group of $C(B)$. (See [16, section 6].) Let $M_s \subset M(B)$ be the set of measures supported on $A_s \subset B$. One can verify using the techniques of [16] that since $s \to A_s$ is Borel, $s \to M_s$ is a Borel field of compact convex subsets of $M(B)$ (where the latter has the w^*-topology, of course). Further α-invariance of A_s readily implies β-invariance of M_s. That is, $\beta^*(s,g)M_{sg} = M_s$ a.e. where $\beta^*(s,g)$ is the adjoint cocycle defined by $(\beta(s,g)^{-1})^*$. It follows from the amenability of S as a G-space that there is a Borel field of measures $s \to \mu_s \in M_s$ such that $\beta^*(s,g)\mu_{sg} = \mu_s$ a.e. Thus, we get a Borel field of Hilbert spaces $s \to H_s = L^2(A_s, \mu_s)$.

Let $K: B \times B \to [0,1]$ be defined by $K(x,y) = \|x-y\|$. Then for each $s \in S$ and integer n, we can define Hilbert–Schmidt operators $T_n(s): H_s \to H_s$ by

$$(T_n(s)f)(x) = \int K(x,y)^n f(y)\,d\mu_s(y).$$

Since K can be considered as a Borel function on $S \times B \times B$ (independent of S), one easily sees that for each n, $T_n(s)$ is a Borel field of self-adjoint Hilbert–Schmidt operators. Since $\alpha(s,g)$ acts as an isometry on B and μ_s is β-invariant, one can readily check that for each n, g, and almost all s,

$$T_n(s)\beta(s,g) = \beta(s,g)T_n(sg),$$

where by β here we mean the induced transformation $L^2(B,\mu_{sg}) \to L^2(B,\mu_s)$, as well as the transformation on $C(B)$. Arguing as in [12, proof of theorem 7.8], we see that the eigenspaces of $T_n(s)$ corresponding to non-zero eigenvalues define G-invariant finite dimensional subbundles of $\int^{\oplus} H_s$. Let $W = \int^{\oplus} W_s \subset \int^{\oplus} H_s$, $W_s \subset H_s$, be chosen so that W is the sum of all G-invariant finite dimensional subbundles. It follows from the above remarks that $W_s^{\perp} \subset \bigcap_n \ker T_n(s)$ a.e.

We now distinguish two cases. Let $b(\mu_s)$ be the barycenter of the measure μ_s, so that $s \to b(\mu_s)$ is a Borel map from S into B. Since μ_s is β-invariant, it is clear that $b(\mu_s)$ is α-invariant. Thus in the case in which $b(\mu_s) \neq 0$ on a set of positive measure (which by ergodicity would imply this was true on a conull set), $s \to \mathbf{C}b(\mu_s)$ defines a G-invariant field of 1-dimensional subspaces, and hence a

1-dimensional subcocycle of α, proving our assertion. Thus, we need only consider the case in which $b(\mu_s)=0$ a.e.

Define a map $Z:H\to C(B)$ by $(Z(w))(x)=\langle x \mid w\rangle$. For each s we have a natural linear map $C(B)\to H_s$ $(=L^2(B,\mu_s))$ and letting $Z_s:H\to H_s$ be the composition of Z with this map, Z_s becomes a bounded Borel field of operators each with norm $\leqq 1$. It is virtually immediate that Z_s is an intertwining field for α and β [11, theorem 2.6], i.e., $Z_s\alpha(s,g)=\beta(s,g)Z_{sg}$ a.e. By the remarks above and the remarks in [11, p. 385], to see that α has a finite dimensional subcocycle, it suffices to show that for s in a set of positive measure, $Z_s(H)$ is not contained in $\bigcap_n \ker T_n(s)$.

Suppose to the contrary that $H_0\subset H$ is a countable dense subset and that for each $f\in H_0$, $Z_s(f)\in\bigcap\ker T_n(s)$ a.e. Let $K_x(y)=K(x,y)$. Then we have, by the definition of $T_n(s)$, $\int Z_s(f)K_x^n d\mu_s=0$ for all n and μ_s-almost all x. Since A_s is norm conpact, the norm and weak topologies coincide. Thus K^n is continuous on $A_s\times A_s$, so that $T_n(s)Z_s(f)$ is continuous on A_s which implies that $\int Z_s(f)K_x^n d\mu_s=0$ for all x in the support of μ_s. Since $b(\mu_s)=0$, we also have $\int Z_s(f)d\mu_s=0$. By Stone–Weierstrass, it follows that $\int Z_s(f)a(K_x)d\mu_s=0$ for any continuous function a defined on $[0,2]$. If for each $f\in H_0$, Z_sf is identically zero on $\operatorname{supp}(\mu_s)$ for almost all s, it would follow that $\operatorname{supp}(\mu_s)=\{0\}$ a.e., which contradicts the fact that $0\not\in A_s$. Thus there is some $f\in H_0$ such that $Z_s(f)$ is not identically zero on $\operatorname{supp}(\mu_s)$ for s in a set of positive measure.

Choose a point $x_0\in\operatorname{supp}(\mu_s)$ on which $Z_s(f)$ achieves a maximum value (for $\operatorname{supp}(\mu_s)$). We can assume $Z_s(f)(x_0)>0$. Choose $r>0$ such that $(Z_sf)(y)>0$ for $y\in\operatorname{supp}(\mu_s)$ with $\|y-x_0\|\leqq r$. Choose a continuous function a on $[0,2]$ such that $a\geqq 0$, $a(0)=1$ and $a(t)=0$ for $t\geqq r$. Since $x_0\in\operatorname{supp}(\mu_s)$, it follows that

$$\int (Z_sf)a(K_{x_0})d\mu_s>0,$$

which is a contradiction. This completes the proof.

We note that the only point at which the assumption of amenability of S as a G-space was used was in showing the existence of the invariant field of measures μ_s. It seems likely that in the present context, such a field would exist for any S. If this is so, the above proof would go through in general.

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